Khovanov homology and four-dimensional topology

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Outline

1. Four-dimensional topology
2. Khovanov homology and its applications
3. A strategy (not yet successful) to construct an exotic 4-sphere

Four dimensions

Four dimensions are special in topology:

- Manifolds of dimension ≤ 3 can be studied using geometric methods (e.g. hyperbolic geometry);
- In dimensions ≥ 5, there is "enough room" to keep disks apart from each other $\longrightarrow$ classification of (simply connected) smooth manifolds using surgery techniques (cf. **Milnor, Smale, Kervaire, Novikov, Wall**, etc. 1960s)
- Dimension 4 is the first where the distinction between *topological* and *smooth* manifolds appears.Topological (simply connected) 4-manifolds have been classified by **Freedman** (1982). Smooth 4-manifolds are still a mystery.

Exotic smooth structures

An exotic smooth structure on a smooth manifold X is given by a manifold X' that is homeomorphic but not diffeomorphic to X.

Example: $\mathbb{R}^n$ has a unique smooth structure for $n \neq 4$, but uncountably many for $n = 4$.

Smooth structures on S^n are unique for $n = 1, 2, 3, 5, 6$; they can be classified for $n \geq 5$ in terms of stable homotopy groups of spheres.

The **smooth Poincaré Conjecture in dimension** 4 **(SPC4)** is still open:

If a smooth 4-manifold X is homotopy equivalent (hence homeomorphic) to S^4, then is it diffeomorphic to S^4?

Opinions are split on whether we should expect this to be true.

Over time, many potential counterexamples have been proposed (manifolds that are homeomorphic to S^4, but not known to be diffeomorphic to it). Many of them were later shown to be standard S^4's.

Gauge theory

Starting with the work of **Donaldson (1982)**, much progress in smooth 4D topology has been made using *gauge theory:* the study of certain PDEs (Yang-Mills, Seiberg-Witten) involving connections and sections of bundles over the 4-manifold, with symmetry under the action of the gauge group of bundle automorphisms.

Here are some major results.

Theorem (Donaldson, 1982)

If the intersection form $Q_X : H_2(X) \times H_2(X) \to \mathbb{Z}$ *of a smooth, simply connected, closed 4-manifold* X *is definite, then* Q_X *can be diagonalized over* $\mathbb{Z}$*.*

A consequence is the existence of non-smoothable 4-manifolds (Freedman's E_8-manifold), and that of exotic smooth structures on $\mathbb{R}^4$.

Applications of gauge theory

Counting solutions to PDEs $\longrightarrow$ invariants of smooth 4-manifolds (Donaldson, Seiberg-Witten). These can distinguish exotic smooth structures.

Since then exotic smooth structures have been found on many closed 4-manifolds, e.g. complex surfaces such as the K3 surface, or $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ for $n \geq 2$ where

$$\mathbb{CP}^2 = \{[z_0 : z_1 : z_2] \in \mathbb{C}^3 \setminus 0\}/[z_0 : z_1 : z_2] \sim [\lambda z_0 : \lambda z_1 : \lambda z_2], \lambda \in \mathbb{C}^*$$

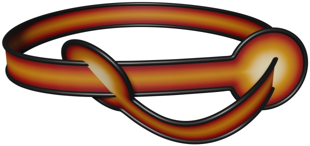
Their existence is still unknown on "small" closed 4-manifolds such as S^4, $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, $S^2 \times S^2$.

Surfaces in B^4

Given a knot $K \subset S^3$, we can ask about its **slice genus**:

$$g_s(K) = \min\{\text{genus}(\Sigma) \mid \Sigma \text{ oriented}, \Sigma \subset B^4, \partial\Sigma = \Sigma \cap S^3 = K\}$$

Knots with $g_s(K) = 0$ are called **slice**. For example:



There is also a weaker notion, of **topologically slice**: bounding a topological disk with a collar neighborhood.

Slice genus

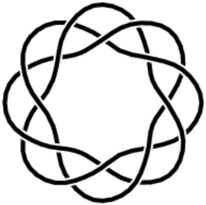
There is no known general algorithm for determining g_s of a knot, or whether a given knot is slice.

However, gauge theory can sometimes help:

Corollary (Milnor conjecture, cf. Kronheimer-Mrowka, 1993)

The slice genus of the torus knot $T_{p,q}$ is $(p-1)(q-1)/2$.

Here, $T_{p,q}$ is obtained by wrapping p strands on a torus, q times; e.g. $(p, q) = (3, -8)$:

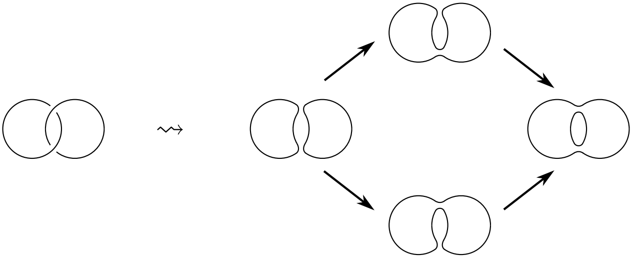


Khovanov homology

For links $K \subset S^3$, **Khovanov** (1999) defined a homology theory

$$Kh(K) = \bigoplus_{i,j} Kh_{i,j}(K).$$

Its construction does not involve PDEs, but rather taking all possible "resolutions" of a link diagram, associating a two-dimensional vector space V to each circle in a resolution, and defining a chain complex using an algebraically-defined differential d:



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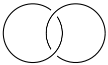
 $\rightsquigarrow$

$$\begin{array}{ccccc} & & V\{1\} & & \\ & \overset{d}{\nearrow} & & \overset{d}{\searrow} & \\ V^{\otimes 2} & & & & V^{\otimes 2}\{2\} \\ & \underset{d}{\searrow} & & \underset{d}{\nearrow} & \\ & & V\{1\} & & \end{array}$$

The differential on Khovanov homology

Let $V = \text{Span } \{1, x\}$.

If two circles get joined together, we use the multiplication map

$$m : V \otimes V \to V$$

$$1^2 = 1,\ 1x = x1 = x,\ x^2 = 0.$$

If one circle gets split in two, we use the comultiplication

$$\Delta : V \otimes V \to V$$

$$\Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x.$$

The homology of the resulting complex is $Kh(K)$.

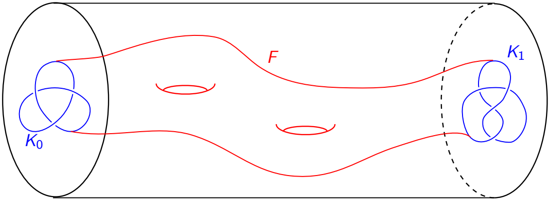
More on Khovanov homology

Khovanov homology was constructed combinatorially, and is closely connected to representation theory (categorification).

Its Euler characteristic is the famous Jones polynomial:

$$\sum_{i,j}(-1)^i q^j \dim Kh_{i,j}(K) = J_K(q)$$

A surface (knot cobordism) $F \subset S^3 \times [0,1]$ from K_0 to K_1 induces a map on Khovanov homology: $Kh(F) : Kh(K_0) \to Kh(K_1)$.



The Rasmussen invariant

We can get $4D$ applications of Khovanov homology related to knots in S^3.

Using a deformation of Khovanov homology, **Rasmussen (2004)** extracted a numerical knot invariant denoted s, which gives a lower bound for the slice genus

$$|s(K)| \leq 2g_s(K).$$

He then used s to give a combinatorial proof of Milnor's Conjecture, that $g_s(T_{p,q}) = (p-1)(q-1)/2$.

One can also use s to show the existence of topologically slice knots that are not smoothly slice, which gives a new proof (without gauge theory) of the existence of exotic smooth structures on $\mathbb{R}^4$.

Still, other gauge-theoretic results (e.g. Donaldson's diagonalizability theorem) do not yet have proofs based on Khovanov homology.

Extensions for some 3-manifolds

There are also extensions of Khovanov homology to knots in a few other closed 3-manifolds: $S^1 \times S^2$ (**Rozansky**), $\#^n(S^1 \times S^2)$ (**Willis**) and $\mathbb{RP}^3$ (**Asaeda-Przytycki-Sikora, Gabrovšek**).

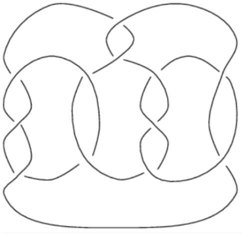
Extensions of the Rasmussen invariant to these settings were constructed by **M.-Marengon-Sarkar-Willis** (for $\#^n(S^1 \times S^2)$) and **M.-Willis** (for $\mathbb{RP}^3$; in progress).

Using these, one can bound the minimal genus of surfaces in $S^1 \times B^3$ or $B^2 \times S^2$ with boundary a knot $K \subset S^1 \times S^2$; or of surfaces in $\mathbb{RP}^3 \times [0,1]$ with boundary a knot in $\mathbb{RP}^3 = \mathbb{RP}^3 \times \{1\}$.

Sample application (M.-Willis): There exist knots K_1, K_2 in $\mathbb{RP}^3 = S^3/\tau$ that are not concordant (they do not co-bound an annulus in $\mathbb{RP}^3 \times [0,1]$), but such that their lifts to S^3 are concordant (co-bound an annulus in $S^3 \times [0,1]$).

A new application (back in S^3)

Up until 2018, it was known which knots with up to 12 crossings are slice, with one exception: *Conway's knot C*.

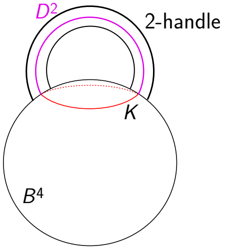


This is topologically slice, has $s(C) = 0$, and many other obstructions to sliceness vanish.

Piccirillo (2018) proved that the Conway knot C is not slice, as follows.

The knot trace

Given a knot K, we can construct a four-manifold $X(K)$ (with boundary), the *trace* of K, by attaching a 2-handle (a neighborhood of a disk) to B^4 along K:



The boundary of $X(K)$ is a 3-manifold called the 0-surgery on K:

$$S^3_0(K) = (S^3 - \text{nbhd}(K)) \cup (S^1 \times D^2),$$

where the gluing reverses the meridian and longitude of the torus $\partial(\text{nbhd}(K)) = S^1 \times S^1$.

New applications

Lemma (Trace Embedding Lemma)

A knot $K \subset S^3$ is slice $\iff$ $X(K)$ smoothly embeds in B^4.

Piccirillo (2018) showed that Conway's knot C is not slice by constructing a partner knot C' such that $X(C) = X(C')$. Then C=slice $\iff$ C' = slice, but $s(C') \neq 0 \Rightarrow C'$ is not slice.

No proof is known using gauge theory.

Can Khovanov homology say something new about 4-manifolds?

Ideally, we would like to use Khovanov homology to construct 4-manifold invariants. **Morrison-Walker-Wedrich (2019)** proposed a candidate, the *skein lasagna algebra*. So far it can only be computed in simple examples like S^4, disk bundles over S^2, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, $S^1 \times S^3$, $S^1 \times B^3$; see **M.-Neithalath (2020)** and **M.-Walker-Wedrich (2021)**.

A possible approach to SPC4

Freedman-Gompf-Morrison-Walker (2009) proposed using Rasmussen's invariant to disprove the smooth $4D$ Poincaré Conjecture:

Find a knot $K \subset S^3$ such that $s(K) \neq 0$ (hence $g_s(K) \neq 0$, i.e. K does not bound a smooth disk in B^4) but K bounds a smooth disk in some homotopy ball Z. Therefore, $Z \ncong B^4$ and $Z \cup B^4$ would be a nontrivial homotopy 4-sphere.

Note: Gauge theoretic invariants cannot distinguish between sliceness in B^4 and in a homotopy 4-ball. It is unclear whether s can do so.

To attempt this strategy, one needs a good source of examples of homotopy 4-balls (or, equivalently, homotopy 4-spheres) and knots that bound disks in them.

I will describe three recent results about the FGMW strategy.

Gluck twists

Gluck (1962): Consider an embedded sphere (2-knot) $S^2 \hookrightarrow S^4$. A neighborhood N of it is diffeomorphic to $S^2 \times D^2$.

Remove N and glue it back:

$$X = (S^4 \setminus N) \cup_f N$$

where

$$f : S^1 \times S^2 \to S^1 \times S^2, \ f(e^{i\theta}, x) = (e^{i\theta}, \mathrm{rot}_\theta(x))$$

The result is a homotopy 4-sphere X. For many families of 2-knots this is known to be diffeomorphic to S^4, but not in general.

A negative result

Theorem (M.-Marengon-Sarkar-Willis, 2019)

If K bounds a smooth disk in a homotopy 4-ball Z obtained from B^4 by a Gluck twist, then $s(K) = 0$. Thus, the FGMW strategy fails for Gluck twists.

Sketch of proof: We show that if K bounds a null-homologous disk in $\mathbb{CP}^2 \# B^4 = \mathbb{CP}^2 \setminus B^4$, then $s(K) \geq 0$. Similarly, if it bounds a null-homologous disk in $\overline{\mathbb{CP}^2} \setminus B^4$, then $s(K) \leq 0$.

On the other hand, it was known that if Z is obtained from B^4 by a Gluck twist, then

$$Z \# \mathbb{CP}^2 \cong B^4 \# \mathbb{CP}^2, \quad Z \# \overline{\mathbb{CP}^2} \cong B^4 \# \overline{\mathbb{CP}^2}.$$

Thus, for K as in the hypothesis, we have $s(K) \geq 0$ and $s(K) \leq 0$. □

A more positive result

An analogue of the FGMW strategy works in other 4-manifolds:

Theorem (M.-Marengon-Piccirillo, 2020)

There exist smooth, closed, homeomorphic four-manifolds X and X' and a knot $K \subset S^3$ that bounds a null-homologous disk in $X \setminus B^4$ but not in $X' \setminus B^4$.

For example, one can take

$$X = \#3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}, \quad X' = K3\#\overline{\mathbb{CP}^2},$$

and K be the trefoil: 

The proof uses gauge theory (the Seiberg-Witten equations).

Another construction of homotopy 4-spheres

Suppose we have two knots K and K' and a homeomorphism

$$\phi : S^3_0(K) \to S^3_0(K').$$

Suppose also that K is slice, bounding a disk $D \subset B^4$. Then $V = B^4 \setminus \text{nbhd}(D)$ has boundary $S^3_0(K)$, the zero-surgery on K. Then

$$W = V \cup_{S^3_0(K)} (-X(K'))$$

is a homotopy 4-sphere. Moreover, K' bounds a disk in W. If we found an example such that K' is not slice (e.g. $s(K') \neq 0$), then SPC4 is false!

Remark: If ϕ extends to a trace diffeomorphism

$$X(K) \xrightarrow{\cong} X(K'),$$

then $W \cong S^4$. We would like to avoid this case.

Knots with the same 0-surgeries

Constructions in the literature:

- dualizable patterns (**Akbulut, Lickorish, Brakes; 1977-80)**;
- annulus twisting (**Osoinach, 2006**);
- some satellites (**Yasui, 2015**).

In some cases these produce knots with the same traces.

M.-Piccirillo (2021) give a general construction of *all* zero-surgery homeomorphisms $\phi : S^3_0(K) \to S^3_0(K')$ based on certain 3-component links called *RBG links*.

We will describe a special case.

Special RBG links

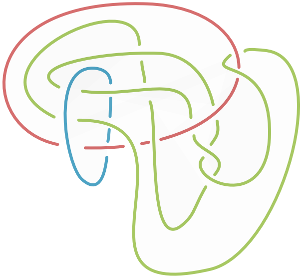
Definition

A *special RBG link* is a 3-component link $L = R \cup B \cup G$ such that there are isotopies

$$R \cup B \cong R \cup \mu_R \cong R \cup G$$

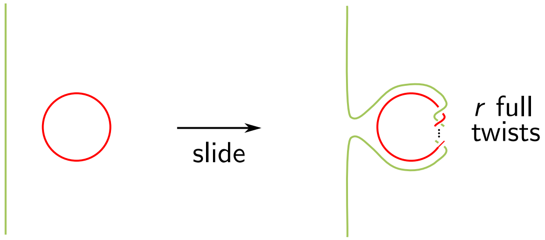
and R is r-framed such that the linking number $l = lk(B, G)$ satisfies $l = 0$ or $rl = 2$. (Here, μ_R is a meridian for R.)

Example:



Slides

From a special RBG link L we obtain a knot K_G by sliding G over R until no geometric linking of B and G remains. Similarly, we obtain a knot K_B by sliding B over R until no geometric linking of B and G remains.

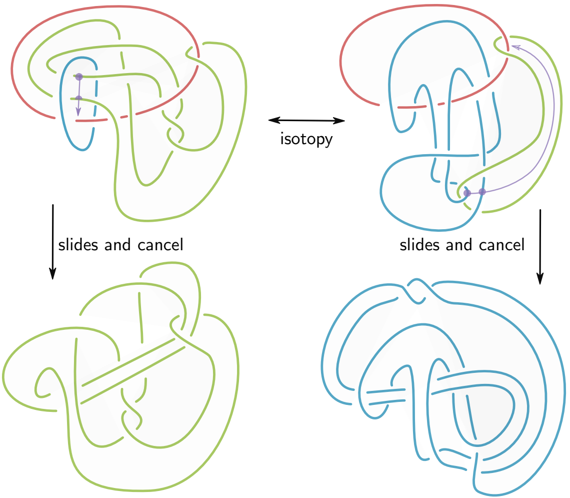


Theorem (M.-Piccirillo, 2021)

If L is a (special) RBG link, there is an associated homeomorphism

$$\phi_L : S^3_0(K_B) \to S^3_0(K_G).$$

An example



Computer experiments

Goal: Find an example where K_B is slice and $s(K_G) \neq 0$ (or vice versa). If V is the complement of a slice disk for K_B, then the homotopy 4-sphere

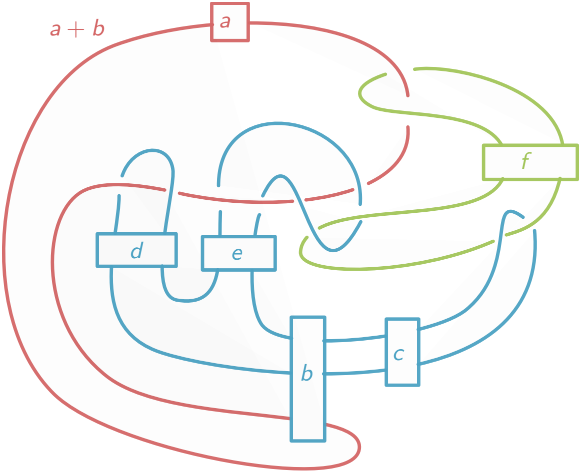
$$W = V \cup_{S^3_0(K)} (-X(K_G))$$

would be exotic, and we would disprove SPC4.

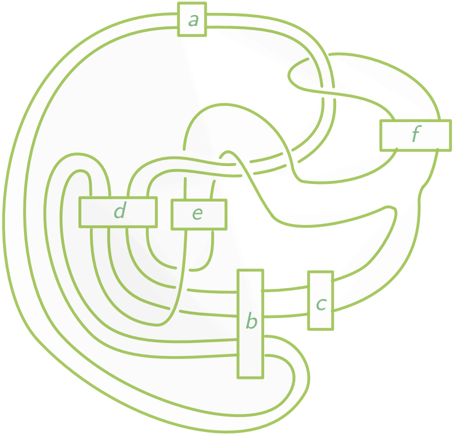
We initially studied a 6-parameter family consisting of 3375 special RBG links. We found no examples as above. In most cases, the knots in the same pair have the same s-invariant.

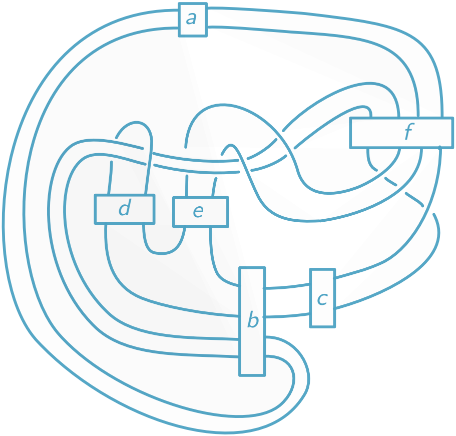
However, in about 1% of cases, the s-invariants differ. In 5 of those examples, one knot has $s \neq 0$, and we could not immediately determine if the other knot was slice.

Here is our family, where the boxes denote the number of full twists.

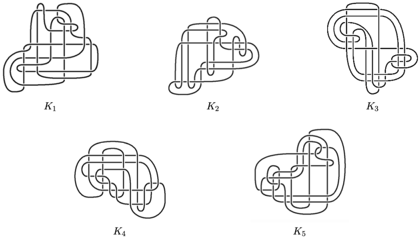


The resulting knots K_B and K_G





Here are 5 topologically slice knots, whose companions (with the same 0-surgery) have $s \neq 0$. If any of them had been slice, then SPC4 would have been false:



However, **Kai Nakamura** later showed they are not slice, and in fact that the s-invariant cannot help disprove SPC4 using special RBG links where R is the unknot.

Looking for slice knots

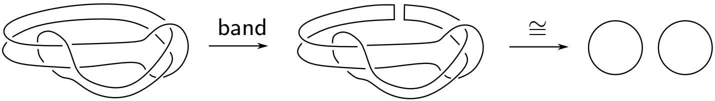
One can still hope to either:

- Consider other RBG links (e.g. special RBG links where R is a nontrivial knot) and search for slice/non-slice pairs using the s invariant; or
- Consider special RBG links where R is the unknot, but distinguish them using other invariants (e.g refined versions of the s-invariant from the Steenrod squares on Khovanov homology, or from other knot homologies).

In any case, in addition to slice obstructions, we need better methods to detect when a knot is slice.

Looking for slice knots

Starting from a knot diagram, one can search for band moves that transform the knot into an unlink (always increasing the number of components). If these exist, the knot is ribbon (and hence slice):



Work in progress by **Dunfield-Gong** and **Gukov-Halverson-M.-Ruehle** $\rightsquigarrow$ computer programs looking for such bands, by trying minimal paths between the segments on the diagram, or trying random longer paths, or improving this with reinforcement learning (in the spirit of AlphaGo).

Looking for slice knots

From our family of 3375 pairs of knots (with the same 0-surgeries), we managed to determine the slice status of all but 5 pairs. (Roughly 25% were slice, and 75% were not.) For the remaining 10 knots, all known obstructions (including s) vanish, but no good bands were found.

In 3 of these pairs, the knots have the same 0-trace so they cannot give counterexamples to SPC4. (If one is slice, then so is the other.) This leaves 2 interesting pairs: